B.Sc. II SEMESTER

Mathematics

PAPER – II

ALGEBRA
AND
GEOMETRY

UNIT-IV

CONE

Syllabus:

Unit – IV

Cone

Equation of a cone, Enveloping cone of a sphere, Ring circular cone.

-10HRS

K.L.E Society's

G.I. Bagewadi Arts, Science & Commerce College, NIPANI

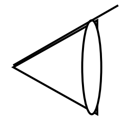
DEPARTMENT OF MATHEMATICS

B.Sc. II Sem. Paper II

Lecture on Cone April 2020

Theorem: General equation of the cone with vertex at origin is homogenous of second degree of the type $ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0$.

Cor.1: If $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a generator for the cone $ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0$ then prove that D.R's must satisfy eqn. of cone.



Proof: Given generator is If $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r$

$\therefore$ any point on the generator is (lr, mr, nr), which lies on the cone

$ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0$---(1)

⇨ This point (lr, mr, nr) must satisfy (1)

$\therefore$ we have $a(lr)^2+b(mr)^2+c(nr)^2+2h(mr)(nr)+2g(nr)(mr)+2f(mr)(lr)=0$

i.e $r^2(a(l)^2+b(m)^2+c(n)^2+2hlm+2gln+2fmn)=0$

But $r^2 \neq 0$, $\therefore$ $al^2+bm^2+cn^2+2hlm+2gln+2fmn=0$

i.e D.R's satisfy the eqn. of cone.

Cor.2: General equation of the cone with vertex at origin and passing through coordinate axis is hxy +gzx +fyz=0.

Proof: Let General equation of the cone with vertex at origin be $ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0$.---(1)

If (1) passes through coordinate axis (they areas generators) then D.R.'s of x-axis, y-axis and z-axis must satisfy eqn. (1) by cor.(1)

But D.R's of x-axis are 1,0,0 , they satisfy eqn. (1) => $a(1)^1+0+0+0+0+0=0$, => a=0

Similarly D.R's of y-axis are 0,1,0 and z-axis 0,0,1 must satisfy (1)

⇨ b=0 and c=0

$\therefore$ eqn. (1) becomes 0+0+0+ 2hxy +2gzx +2fyz=0.

i.e hxy +gzx +fyz=0.

Thus equation of the cone with vertex at origin and passing through coordinate axis is hxy +gzx +fyz=0.

Examples:

1. **Prove that the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, where $l^2+2m^2-3n^2=0$ is a generator for the cone $x^2+2y^2-3z^2=0$.**

   Proof: We know that if $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a generator for the cone $x^2+2y^2-3z^2=0$ then by cor. (1), D.R's l, m, n of the generator must satisfy the eqn of cone.

   $\therefore$ we have $l^2+2m^2-3n^2=0$ which is true.

2. **Find eqn. of cone generated by the line through (1,2,3)whose D.C.'s satisfy the eqn. $2l^2+3m^2-4n^2 = 0$.**

Soln.: Eqn of generator passing through the point (1,2,3) with D.C's l, m, n is

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = r \text{ (say)}$$

⇨ $l = \frac{x-1}{r}$, $m = \frac{y-2}{r}$, $n = \frac{z-3}{r}$

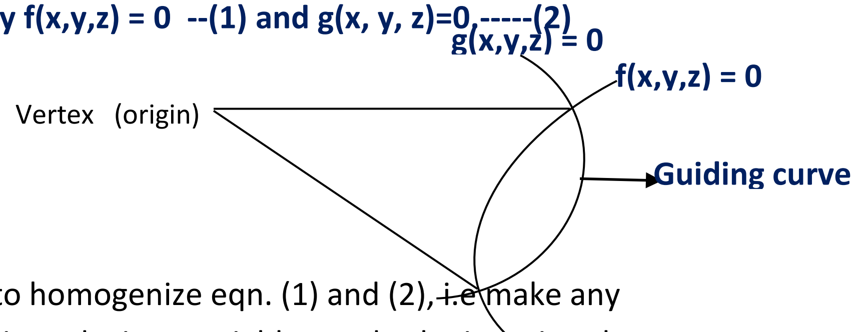
By cor. (1) we know that D. C's of generator satisfy the eqn. of cone and that eqn. is given by $2l^2+3m^2-4n^2 = 0$ -----------(1)

Substitute l, m, n in (1) we get $2\left(\frac{x-1}{r}\right)^2+3\left(\frac{y-2}{r}\right)^2-4\left(\frac{z-3}{r}\right)^2 = 0$

i.e $2(x-1)^2 + 3(y-2)^2 -4(z-3)^2 = 0$.

i.e $2x^2 + 3y^2 - 4z^2 -4x - 12y+24z -22=0$ which is required eqn. of cone.

Finding eqn of cone with vertex at origin and passing through the guiding curve as intersection of curves given by f(x,y,z) = 0 --(1) and g(x, y, z)=0 -----(2)



To find eqn. of cone we have to homogenize eqn. (1) and (2), i.e make any One eqn. as homogeneous by introducing variable t and substitute in other.

We come to know by following examples: (these examples are important for 2 marks)

3. Find Find eqn. to the cone with vertex at origin which passes through the curve given by $ax^2+by^2+cz^2 = 1$ and $\alpha x^2 + \beta y^2 = 2z$.

Soln.: Given curve is intersection of $ax^2+by^2+cz^2 = 1$ --------(1)

$\alpha x^2 + \beta y^2 = 2z$.------(2)

We have make both homogeneous by introducing 3rd variable 't'

$ax^2+by^2+cz^2 = t^2$ -------(3) homo. of degree 2.

$\alpha x^2 + \beta y^2 = 2zt$.---------(4) homo. of degree 2.

From (4), $t = \frac{\alpha x^2 + \beta y^2}{2z}$, substitute this t in (3) we get $ax^2+by^2+cz^2 = \left(\frac{\alpha x^2 + \beta y^2}{2z}\right)^2$

i.e , $4z^2 (ax^2+by^2+cz^2) = (\alpha x^2 + \beta y^2)^2$ which is the Req. eqn. of the cone.

4. Find eqn. to the cone with vertex at origin which passes through the curve given by $x^2+y^2+z^2 + x- 2y +3z -4 = 0$ and $x^2+y^2+z^2 + 2x- 3y +4z - 5 = 0$

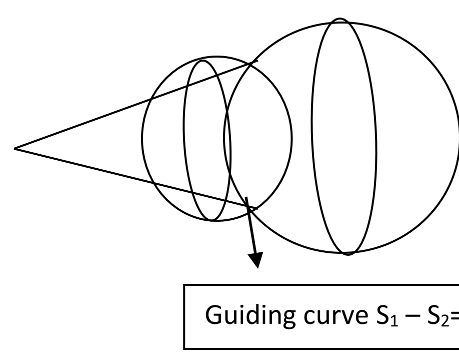
Soln.:. The curve of intersection of

$S_1 : x^2+y^2+z^2 + x- 2y +3z -4 = 0$ --------(1)

$S_2 : x^2+y^2+z^2 + 2x- 3y +4z - 5 = 0$-------(2)

is $S_1 - S_2 = 0$

i.e $x- y +z =1$ ----(3)



Homogenizing (1) and (3) we get, $x^2+ y^2+z^2 + xt- 2yt +3zt -4t^2 = 0$ ----(4) (b'cz this eqn. of degree 2)

& $x- y +z =t$ --------------------------------(5) (b'cz this eqn. of degree 1)

Substitute t=x-y+z in (4) we get $x^2+ y^2+z^2 +(x- 2y +3z)(x-y+z) -4(x-y+z)^2 = 0$ req.eqn.

5. Find eqn. to the cone with vertex at origin and base as a circle x=a , $y^2+z^2 = b^2$. (this is a guiding curve, circle lies on the yz plane)

Soln.: The guiding curve is intersection of x=a------------(1)

and $y^2+z^2 = b^2$-------(2)

Homogenizing (1) and (2) we get, x=at => $t = \frac{x}{a}$ ------(3)

and $y^2+z^2 = b^2 t^2$-------(4)

Substitute (3) in (4) we get $y^2+z^2 = b^2 (\frac{x}{a})^2$

i.e $a^2(y^2+z^2) = b^2x^2$, which is req. eqn. of cone.

6. Find eqn. to the cone with vertex at origin and base is $x^2+y^2 = 4$ and z=2. (this is a guiding curve, circle lies on the xy plane)

Try this same as example (5)

7. Find eqn. to the cone with vertex at (0,0,0) which passes through the curve of intersection of $x^2+ y^2+z^2 + x- 2y +3z -4 = 0$ and x-y+z=2.

Try this also. Homogenize these two eqns. And replace t between them we get.

8. Find eqn. to the cone with vertex at (0,0,0) which passes through the curve of intersection of plane lx+my+nz=p and $ax^2+b y^2+cz^2 = 1$.

Soln.: Guiding curve is intersection of lx+my+nz=p ----------(1)

and $ax^2+b y^2+cz^2 = 1$.-----(2)

Homogenizing (1) and (2) we get, lx+my+nz=pt --(3) (b'cz this eqn. of degree 1)

$ax^2+b y^2+cz^2 = t^2$ ---(4) (b'cz this eqn. of degree 2)

Substitute $t = \frac{lx+my+nz}{p}$ from (3) in (4) we get $ax^2+by^2+cz^2 = (\frac{lx+my+nz}{p})^2$

i.e $p^2(ax^2+by^2+cz^2) = (lx + my + nz)^2$, which is req. eqn. of cone

9. Find eqn. to the cone with vertex at (0,0,0) which passes through the curve of intersection of $ax^2+by^2 = 2z$ plane lx+my+nz=p .

Try this example.

10. Find eqn. to the cone with vertex at (0,0,0) which contains the curve given by(guiding curve) $x^2 - y^2 + 4ax=0$ plane x+y+z=6 .

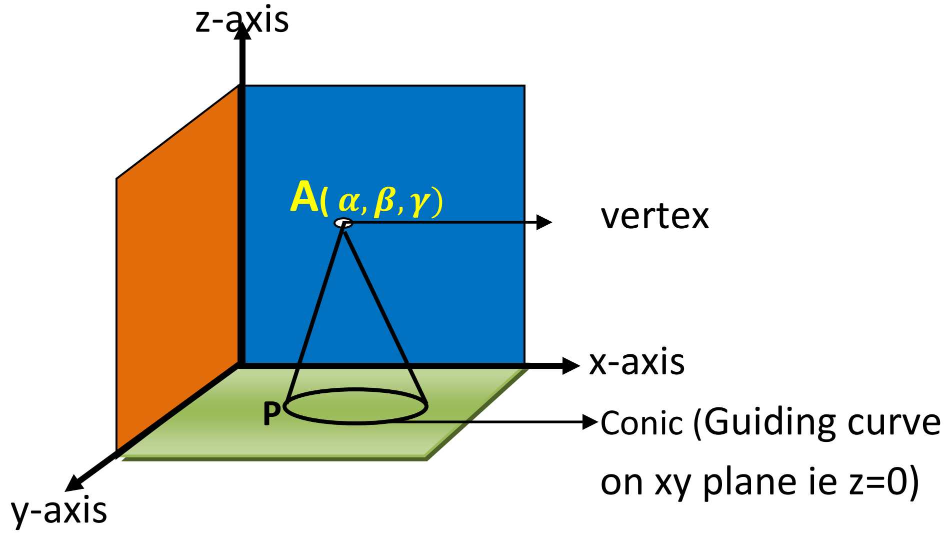
Hint: Homogenizing, $x^2 - y^2 + 4axt=0$ and x+y+z=6t => $t = \frac{x+m+z}{6}$, put in

Theorem: To find eqn. of cone with given vertex (α, β, γ) and conic as guiding curve.

Proof: Let A(α, β, γ) be vertex of the cone and let $ax^2+by^2+2hxy+2gx+2fy+c=0$: z=0 -------(1)

be given conic which is guiding curve.

[i.e guiding curve is conic (it may be ellipse or circle) lies on xy plane i.e z=0, thatswhy in the conic z term is not there. Similarly, if conic lies on yz plane , i.e x=0, then conic contains noly y and z terms and so on.]



Any line **AP** through A($\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$) having D.R.'s l, m, n can be written as $\frac{x-\boldsymbol{\alpha}}{l} = \frac{y-\boldsymbol{\beta}}{m} = \frac{z-\boldsymbol{\gamma}}{n}$ **-------(2)**

But this line intersect xy-plane at z=0 (b'cz on xy plane z coordinate is zero, its eqn. is z=0)

$\therefore$ **we have** $\frac{x-\boldsymbol{\alpha}}{l} = \frac{y-\boldsymbol{\beta}}{m} = \frac{0-\boldsymbol{\gamma}}{n}$

⇨ $\frac{x-\boldsymbol{\alpha}}{l} = \frac{0-\boldsymbol{\gamma}}{n}$ and $\frac{y-\boldsymbol{\beta}}{m} = \frac{0-\boldsymbol{\gamma}}{n}$

⇨ x = $\boldsymbol{\alpha}$ **-** $\frac{l\gamma}{n}$, y= $\boldsymbol{\beta}$ **-** $\frac{m\boldsymbol{\gamma}}{n}$, z = 0, are coordinates of P, which lies on conic (1), as shown in fig.

substitute in (1) we get ,

$a(\alpha - \frac{l\gamma}{n})^2 + + 2h(\alpha - \frac{l\gamma}{n})(\beta - \frac{m\gamma}{n}) + b(\beta - \frac{m\gamma}{n})^2 + 2g(\alpha - \frac{l\gamma}{n}) + 2f(\beta - \frac{m\gamma}{n}) + c = 0$

i.e $a(\alpha - \gamma\frac{l}{n})^2 + + 2h(\alpha - \gamma\frac{l}{n})(\beta - \gamma\frac{m}{n}) + b(\beta - \gamma\frac{m}{n})^2 + 2g(\alpha - \gamma\frac{l}{n}) + 2f(\beta - \gamma\frac{m}{n}) + c = 0$**---(3)**

But from (2) $\frac{x-\boldsymbol{\alpha}}{l} = \frac{z-\boldsymbol{\gamma}}{n}$ and $\frac{y-\boldsymbol{\beta}}{m} = \frac{z-\boldsymbol{\gamma}}{n}$

⇨ $\frac{x-\boldsymbol{\alpha}}{z-\boldsymbol{\gamma}} = \frac{l}{n}$ and $\frac{y-\boldsymbol{\beta}}{z-\boldsymbol{\gamma}} = \frac{m}{n}$ i.e $\frac{l}{n} = \frac{x-\boldsymbol{\alpha}}{z-\boldsymbol{\gamma}}$ and $\frac{m}{n} = \frac{y-\boldsymbol{\beta}}{z-\boldsymbol{\gamma}}$

Substitute these values in (3) we get

$a(\alpha - \gamma\frac{x-\boldsymbol{\alpha}}{z-\boldsymbol{\gamma}})^2 + 2h(\alpha - \gamma\frac{x-\boldsymbol{\alpha}}{z-\boldsymbol{\gamma}})(\beta - \gamma\frac{y-\boldsymbol{\beta}}{z-\boldsymbol{\gamma}}) + b(\beta - \gamma\frac{y-\boldsymbol{\beta}}{z-\boldsymbol{\gamma}})^2 + 2g(\alpha - \gamma\frac{x-\boldsymbol{\alpha}}{z-\boldsymbol{\gamma}}) + 2f(\beta - \gamma\frac{y-\boldsymbol{\beta}}{z-\boldsymbol{\gamma}}) + c = 0$

LCM is $(z - \boldsymbol{\gamma})^2$,

∴ **we get**

$$a[\alpha (z-\gamma)-\gamma (x-\alpha)]^2+2h[\alpha (z-\gamma)-\gamma (x-\alpha)]\,[\beta (z-\gamma)-\gamma (y-\beta)]$$
$$+b\,[\beta (z-\gamma)-\gamma (y-\beta)]^2+2g\,[\alpha (z-\gamma)-\gamma (x-\alpha)](z-\gamma)$$
$$+2f[\beta (z-\gamma)-\gamma (y-\beta)]\,(z-\gamma)+c\,(z-\gamma)^2=0$$

Simplifying we get

$$a(\alpha z-\gamma x)^2+2h(\alpha z-\gamma x)(\beta z-\gamma y)+b(\beta z-\gamma y)^2$$
$$+2g(\alpha z-\gamma x)\,(z-\gamma)+2f(\beta z-\gamma y)(z-\gamma)+c\,(z-\gamma)^2=0$$

Which is req. eqn. of cone with vertex as A$(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$.

NOTE: Remember the procedure which we are applying for examples, not direct formula, whatever the theory is there same we have to apply for example.

Examples:

1. Find eqn. to the cone with vertex at (1,2,3) and whose generating line pass through the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and z=0 .(i.e guiding curve is ellipse lies on xy plane)

Soln.: Given conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and z=0 ------------------(1)

Any line **AP** through A$(\mathbf{1},\mathbf{2},\mathbf{3})$ having D.R.'s l, m, n can be written as $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}$ **-------(2)**

But this line intersect xy-plane at z=0 (b'cz on xy plane z coordinate is zero, its eqn. is z=0)

∴ **we have** $\frac{x-1}{l}=\frac{y-2}{m}=\frac{0-3}{n}$

⇨ $\frac{x-1}{l}=\frac{0-3}{n}$ and $\frac{y-2}{m}=\frac{0-3}{n}$

⇨ $x=\mathbf{1}-\mathbf{3}\frac{l}{n}$, $y=\mathbf{2}-\mathbf{3}\frac{m}{n}$, z = 0, are coordinates of P, which lies on conic (1),

substitute in (1) we get ,

$\frac{(1-3\frac{l}{n})^2}{a^2}+\frac{(2-3\frac{m}{n})^2}{b^2}=\mathbf{1}$ **---(3)**

But from (2) $\frac{x-1}{l}=\frac{z-3}{n}$ and $\frac{y-2}{m}=\frac{z-3}{n}$

⇨ $\frac{x-1}{z-3}=\frac{l}{n}$ and $\frac{y-2}{z-3}=\frac{m}{n}$ i.e $\frac{l}{n}=\frac{x-1}{z-3}$ and $\frac{m}{n}=\frac{y-2}{z-3}$

Substitute these values in (3) we get

$\frac{1}{a^2}(1-\mathbf{3}\frac{x-1}{z-3})^2+\frac{1}{b^2}(\mathbf{2}-\mathbf{3}\frac{y-2}{z-3})^2=1$

LCM is $(z-\mathbf{3})^2$,

∴ **we get**

$\frac{1}{a^2}[(z-3)-\mathbf{3}(x-1)]^2+\frac{1}{b^2}[2(z-3)-3(y-2)]^2=(z-3)^2$

Simplifying we get, $b^2[z-3x]^2+a^2[2z-3y]^2=a^2b^2(z-3)^2$

which is req. eqn. of cone with vertex at A$(\mathbf{1},\mathbf{2},\mathbf{3})$.

2. Find eqn. to the cone with vertex at (0,0,3) and guiding curve $x^2+ y^2 =4$ and z=0 .(i.e guiding curve is circle lies on xy plane)

Soln.: Given conic $x^2+ y^2 =4$ and z=0 ------------------(1)

Any line **AP** through A$(\mathbf{0},\mathbf{0},\mathbf{3})$ having D.R.'s l, m, n can be written as $\frac{x-\mathbf{0}}{l} = \frac{y-\mathbf{0}}{m} = \frac{z-\mathbf{3}}{n}$ **-------(2)**

But this line intersect xy-plane at z=0 (b'cz on xy plane z coordinate is zero, its eqn. is z=0)

∴ **we have** $\frac{x-\mathbf{0}}{l} = \frac{y-\mathbf{0}}{m} = \frac{0-\mathbf{3}}{n}$

⇨ $\frac{x-\mathbf{0}}{l} = \frac{0-\mathbf{3}}{n}$ and $\frac{y-\mathbf{0}}{m} = \frac{0-\mathbf{3}}{n}$

⇨ x = $-\mathbf{3}\frac{l}{n}$, y= -- $\mathbf{3}\frac{m}{n}$, z = 0, are coordinates of P, which lies on conic (1),

substitute in (1) we get ,

$(-\mathbf{3}\frac{l}{n})^2 +(-\ \mathbf{3}\frac{m}{n})^2 = 4$ --**(3)**

But from (2) $\frac{x}{l} = \frac{z-\mathbf{3}}{n}$ and $\frac{y}{m} = \frac{z-\mathbf{3}}{n}$

⇨ $\frac{x}{z-\mathbf{3}} = \frac{l}{n}$ and $\frac{y}{z-\mathbf{3}} = \frac{m}{n}$ i.e $\frac{l}{n} = \frac{x}{z-\mathbf{3}}$ and $\frac{m}{n} = \frac{y}{z-\mathbf{3}}$

Substitute these values in (3) we get

$(-\mathbf{3}\frac{x}{z-\mathbf{3}})^2 + (-\ \mathbf{3}\frac{y}{z-\mathbf{3}})^2 = 4$

LCM is $(z-\mathbf{3})^2$,

∴ **we get**

$[-\ \mathbf{3}(x)]^2 + [-3(y)]^2= 4(z-3)^2$

Simplifying we get, $9x^2+9y^2 = 4(z-3)^2$

which is req. eqn. of cone with vertex at A$(\mathbf{1},\mathbf{2},\mathbf{3})$.

3. Find eqn. to the cone with vertex at (0,0,1) and guiding curve $x^2+ y^2 =1$ and z=0 .(i.e guiding curve is circle lies on xy plane)
4. Find eqn. to the cone with vertex at (1,2,3) and guiding curve $x^2+ y^2 =9$ and z=0 .(i.e guiding curve is circle lies on xy plane)

   Try these two examples
5. Find eqn. to the cone with vertex at (1,1,0) and guiding curve $x^2+ z^2 =4$ and y=0 .(i.e guiding curve is circle lies on xz plane)

Soln.: Given conic $x^2+ z^2 =4$ and y=0 ------------------(1)

Any line **AP** through A$(\mathbf{1},\mathbf{1},\mathbf{0})$ having D.R.'s l, m, n can be written as $\frac{x-\mathbf{1}}{l} = \frac{y-\mathbf{1}}{m} = \frac{z-\mathbf{0}}{n}$ **-------(2)**

But this line intersect xz-plane at y=0 (b'cz on xz plane y coordinate is zero, its eqn. is y=0)

∴ **we have** $\frac{x-\mathbf{1}}{l} = \frac{0-\mathbf{1}}{m} = \frac{z-\mathbf{0}}{n}$

⇨ $\frac{x-\mathbf{1}}{l} = \frac{0-\mathbf{1}}{m}$ and $\frac{z-\mathbf{0}}{n} = \frac{0-\mathbf{1}}{m}$

⇨ x = $\mathbf{1}-\frac{l}{m}$, z= $\frac{-n}{m}$, y = 0, i. e x = $1-\frac{l}{m}$, y = 0, z= $\frac{-n}{m}$, are coordinates of P, which lies on conic (1),

substitute in (1) we get ,

$(\mathbf{1}-\frac{l}{m})^2+(\frac{-n}{m})^2 = 4$ --**(3)**

But from (2) $\frac{x-\mathbf{1}}{l} = \frac{y-\mathbf{1}}{m}$ and $\frac{y-\mathbf{1}}{m} = \frac{z-\mathbf{0}}{n}$

⇨ $\frac{x-1}{y-1} = \frac{l}{m}$ and $\frac{z}{y-1} = \frac{n}{m}$ i.e $\frac{l}{m} = \frac{x-1}{y-1}$ and $\frac{n}{m} = \frac{z}{y-1}$

Substitute these values in (3) we get

$(1-\frac{x-1}{y-1})^2+(-\frac{z}{y-1})^2 = 4$

LCM is $(y-1)^2$,

∴ we get

$[y-x]^2 + [-z]^2 = 4(y-1)^2$

Simplifying we get, $x^2+y^2-2xy+z^2 = 4(y-1)^2$, i.e $x^2-3y^2+z^2-2xy+8y-4=0$

which is req. eqn. of cone with vertex at A$(\mathbf{1}, \mathbf{1}, \mathbf{0})$.

Another type of examples:(Important for 5 marks)

In this type they will give one eqn. and ask to prove it is cone and also to find vertex.

Procedure:

Given eqn.is interms of x,y and z, hence let it be f(x,y,z)=0, which is general eqn. of second degree.

Introduce variable 't' and make the eqn. as homogeneous of second degree in x,y,z and t

Let it be F(x, y, z, t)=0-----------------------------------(1)

Differentiate (1) partially w.r.t x, y, z, and t,

We get four equations namely $\frac{\partial F}{\partial x} = 0$ ------------(2), $\frac{\partial F}{\partial y} = 0$ ------------(3), $\frac{\partial F}{\partial z} = 0$ ------------(4), $\frac{\partial F}{\partial t} = 0$ ------------(5), in all these put t=1 first and solve any three of the above eqns. For x, y and z and substitute in the remaining eqn. it must be satisfied then we say give represent cone with (x,y,z)as vertex.

You come to know by following example

Examples:

1. Prove that eqn. $4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$ represents cone & find its vertex.

Soln.: Given eqn. be $f(x, y, z) = 4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$**--------------(1)**

Make it homogeneous of second degree by introducing t

i.e. $F(x, y, z, t) = 4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2=0$-----------------------------**(2)**

Differentiate (2) partially w.r.t x, y, z, and t,

We get $\frac{\partial F}{\partial x} = 0$, 8x -0 + 0 + 2y +0 + 12t – 0 +0 +0 +0=0 i.e $\frac{\partial F}{\partial x}$ = 0 => 8x + 2y +12t =0, i.e 8x + 2y +12 =0 ---------------------(3) i.e put t=1

Next, $\frac{\partial F}{\partial y}$ = 0 => -2y +2x -3z - 11t =0 , i.e 2x-2y – 3z -11 = 0------------------(4) by sub. t=1

$\frac{\partial F}{\partial z}$ = 0 => 4z – 3y + 6t = 0, i.e 4z – 3y + 6 = 0---------------------(5), t=1

$\frac{\partial F}{\partial t}$ = 0 => 12x -11y +6z +8t=0 i.e 12x -11y +6z +8=0 -------------(6), t=1

(3) – 4(4) gives 8x + 2y +12- (8x – 8y -12z -44) =0

i.e 10y +12z +56=0 i.e 6z + 5y +23=0-------------------------------------(7)

Next, 3(4) – 2(7) gives

12z – 9y +18 –(12z+10y+56)=0

i.e -17y -34 =0 => **y = -2**

Put y = - 2 in (5) we get, 4z – 3(-2) + 6= 0 i.e **z = -3**, and from (3) we get **x =-1**

We used only eqns, (3), (4) and (5) and got **x =-1, y = -2, z = -3** , substitute these in (6)

12(-1) – 11(-2) + 6(-3) +8 = -12+22-18+8 = -30 +30 =0, so it is satisfied by these values.

⇨ Eqn. (1) represents cone with **(-1, -2, -3)** as vertex.

2. Prove that eqn. $2x^2 +2y^2 +7z^2 - 10yz - 10zx +2x +2y +26z -9 =0$ represents cone & find its vertex.

Soln.: Given eqn. be f(x, y, z) = $2x^2 +2y^2 +7z^2 - 10yz - 10zx +2x +2y +26z -17 =0$ **--------------(1**)

Make it homogeneous of second degree by introducing t

i.e. F(x, y, z, t) =$2x^2 +2y^2 +7z^2 - 10yz - 10zx +2xt +2yt +26zt -17t^2 =0$ ---------------------------**(2)**

Differentiate (2) partially w.r.t x, y, z, and t,

We get $\frac{\partial F}{\partial x}$ = 0 , 4x -10z +2t =0 i.e 4x -10z +2 =0 (put t=1) -----------(3)

Next, $\frac{\partial F}{\partial y}$ = 0 => 4y -10z +2t=0 , i.e 4y -10z +2=0----------------------(4) by sub. t=1

$\frac{\partial F}{\partial z}$ = 0 => 14z – 10y - 10x+ 26t = 0, i.e 14z – 10y - 10x+ 26 = 0---------------------(5), t=1

$\frac{\partial F}{\partial t}$ = 0 => 2x +2y +26z -34t=0 i.e 2x +2y +26z - 34=0 -------------(6), t=1

(3) – (4) gives 4x -10z +2 – (4y-10z+2)=0

i.e 4x – 4y =0 ie x – y =0 i.e x =y ------------------------------------(7)

Put x =y in (5) , we get 14z – 10y - 10y+ 26 = 0

Ie 14z -20y +26 =0 i.e -10y + 7z +13=0--------------------(8)

Then 5(4) + 2(8) gives 20y -50z +10 +(-20y +14z +26)=0

-36z +36=0 => z = 1

Put z =1 in (4) we get 4y – 10 +2 =0 => y =2 => x =2 from (7)

Put x=2, y = 2 and z=1 in (6) we get

2(2)+2(2)+26(1)-18=4 + 4 +26 - 34 =34-34 =0

$\therefore$ (6) is satisfied by (2,2,1)

⇨ Eqn. (1) represents cone with **(2,2,1)** as vertex.

3. Prove that eqn. $2x^2 - 8xy - 4yz - 4x - 2y + 6z + 35 = 0$ represents cone & find its vertex.

Soln.: Given eqn. be $f(x, y, z) = 2x^2 - 8xy - 4yz - 4x - 2y + 6z + 35 = 0$ **--------------(1)**

Make it homogeneous of second degree by introducing t

i.e. $F(x, y, z, t) = 2x^2 - 8xy - 4yz - 4xt - 2yt + 6zt + 35t^2 = 0$ ----------------------------**(2)**

Differentiate (2) partially w.r.t x, y, z, and t,

We get $\frac{\partial F}{\partial x} = 0$ => 4x -8y -4t=0 i.e 4x -8y -4 =0 (put t=1) -----------(3)

Next, $\frac{\partial F}{\partial y} = 0$ => - 8x -4z -2t=0 , i.e 8x +4z +2=0----------------------(4) by sub. t=1

$\frac{\partial F}{\partial z} = 0$ => - 4y + 6= 0, => $y = \frac{3}{2}$

$\frac{\partial F}{\partial t} = 0$ => - 4x -2y +6z +70t=0 i.e - 4x -2y +6z +70=0 -------------(6), t=1

Put $y = \frac{3}{2}$ in (3) we get $4x - 8(\frac{3}{2}) - 4 = 0$, => x = 4

Put x=4 in (5) we get, 32+4z+2=0 => 4z = -34 $\therefore z = -\frac{17}{2}$

Put x=4, $y = \frac{3}{2}$ and $z = \frac{-17}{2}$ in (6) we get

$-16 - 2(\frac{3}{2}) + 6(-\frac{17}{2}) + 70 = -16 - 3 - 51 + 70 = -70 + 70 = 0$ $\therefore$ (6) is satisfied by $(4, \frac{3}{2}, -\frac{17}{2})$

⇨ Eqn. (1) represents cone with $(4, \frac{3}{2}, -\frac{17}{2})$ as vertex.

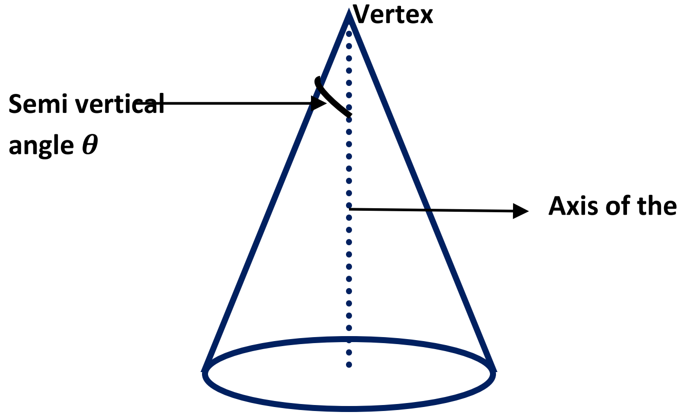
4. Prove that eqn. $x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$ represents cone & find its vertex.

5. Prove that eqn. $2y^2 - 8xy - 8yz - 4zx + 6x - 4y - 2z + 5 = 0$ represents cone & find its vertex.

Try (4) and (5) as home work

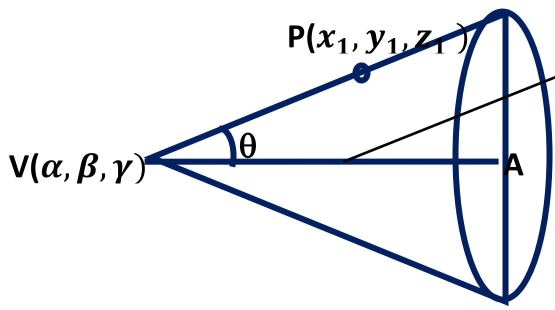
Right circular Cone:

Definition(important for 2. Marks): A surface generated by a line passing through a fixed point and making constant angle with fixed line through vertex is called a right circular cone. The fixed point is called as the vertex, fixed line is called axis of the cone and constant angle is called semi vertical angle of the cone (as shown in fig.)



Theorem: To find the eqn. of the right circular con with vertex V(α, β, γ) , eqn. of the axis as $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and semi vertical angle θ

Proof:



Let P(x_1, y_1, z_1) be any point on the surface of the cone (i.e anywhere on the surface) then VP is the generator of the cone which makes an angle θ with axis of the cone VA.

D.R.'s of VP are $(x_1 - \alpha), (y_1 - \beta), (z_1 - \gamma)$,

D.R.'s of VA are l, m, n and angle between VP and VA is θ

By using the formula for angle between the lines,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n}{\sqrt{(x_1-\alpha)^2 + (y_1-\beta)^2 + ((z_1-\gamma)^2}\sqrt{l^2+m^2+n^2}}$$

$$\therefore \cos\theta = \frac{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n}{\sqrt{(x_1-\alpha)^2 + (y_1-\beta)^2 + ((z_1-\gamma)^2}\sqrt{l^2+m^2+n^2}}$$

Squaring both the sides we get

$$\cos^2\theta = \frac{[(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)\text{n}\,]^2}{\left[(x_1-\alpha)^2+(y_1-\beta)^2+\left((z_1-\gamma)\right)^2\right][l^2+m^2+n^2]}$$

i.e $\cos^2\theta[(\boldsymbol{x_1}-\boldsymbol{\alpha})^2+(\boldsymbol{y_1}-\boldsymbol{\beta})^2+((\boldsymbol{z_1}-\boldsymbol{\gamma})^2][l^2+m^2+n^2] = [(\boldsymbol{x_1}-\boldsymbol{\alpha})\boldsymbol{l}+(\boldsymbol{y_1}-\boldsymbol{\beta})\boldsymbol{m}+(\boldsymbol{z_1}-\boldsymbol{\gamma})\text{n}\,]^2$

But P(x_1, y_1, z_1) be any point on the surface of the cone and hence locus of the point P i.e replace (x_1, y_1, z_1) by (x, y, z) in the above eqn. we get

$\cos^2\theta[(\boldsymbol{x}-\boldsymbol{\alpha})^2+(\boldsymbol{y}-\boldsymbol{\beta})^2+(\boldsymbol{z}-\boldsymbol{\gamma})^2][l^2+m^2+n^2] = [\boldsymbol{x}-\boldsymbol{\alpha})\boldsymbol{l}+(\boldsymbol{y}-\boldsymbol{\beta})\boldsymbol{m}+(\boldsymbol{z}-\boldsymbol{\gamma})\text{n}\,]^2$

Thus the eqn. of the right circular cone with vertex V(α, β, γ), eqn. of the axis as $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and semi vertical angle θ is

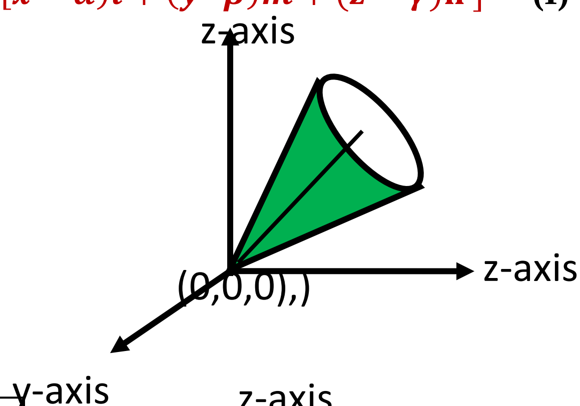
$\boldsymbol{\cos^2\theta[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2][l^2+m^2+n^2] = [x-\alpha)l+(y-\beta)m+(z-\gamma)n\,]^2}$ **--(I)**

Corollary 1: The eqn. of the right circular cone with vertex V(0, 0, 0), eqn. of the axis as $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and semi vertical angle θ is

$\cos^2\theta[x^2+y^2+z^2][l^2+m^2+n^2] = [\text{x}l+ym+zn\,]^2$

Proof: In (I) if we put $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ as (0,0,0) we get

$\cos^2\theta[x^2+y^2+z^2][l^2+m^2+n^2] = [\text{x}l+ym+zn\,]^2$



Corollary 2: The eqn. of the right circular cone with vertex V(0, 0, 0), the axis as $z-axis$ and semi vertical angle θ is $\mathbf{z^2\tan^2\theta = x^2+y^2}$ **(as in fig.)**

Proof: If Z –axis is axis of cone , its D.R.'s 0,0,1,

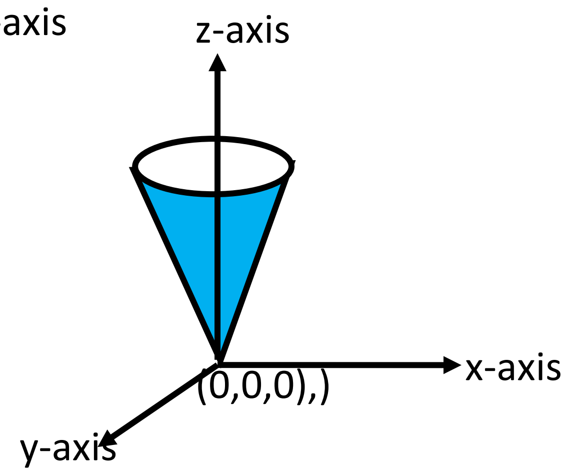
∴ In (I) $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ as (0,0,0) and l, m, n, as 0,0,1

We get, $\cos^2\theta[x^2+y^2+z^2][0+0+1] = [0+0+z\,]^2$

i.e $\cos^2\theta[x^2+y^2] = z^2[1 - \cos^2\theta]$

i.e $\cos^2\theta[x^2+y^2] = z^2\sin^2\theta$

i.e $[x^2+y^2] = z^2\tan^2\theta$ or $z^2\tan^2\theta = x^2+y^2$



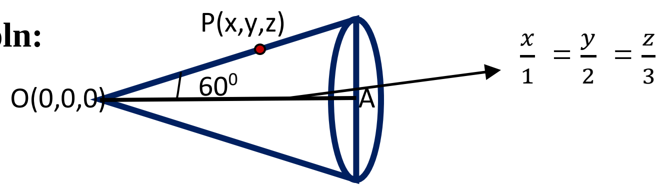
Similarly eqn. of the right circular cone with vertex V(0, 0, 0), the axis as $x-axis$ and semi vertical angle θ is $\mathbf{x^2\tan^2\theta = y^2+z^2}$ **and** eqn. of the right circular cone with vertex V(0, 0, 0), the axis as $y-axis$ and semi vertical angle θ is $\mathbf{y^2\tan^2\theta = x^2+z^2}$

Note: These corollaries are important for 2 marks.

Examples:

1. Find the eqn. of the right circular cone with vertex $(0, 0, 0)$, eqn. of the axis as $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and semi vertical angle θ is 60^0.

Soln:



Soln: Let P(x,y,z) be any point on the surface of the cone.

D.R.'s of OP are x-0, y-0, z-0,

D.R's of axis are 1, 2, 3 and θ =60^0

$\therefore$ cos60 = $\frac{x1+y2+z3}{\sqrt{x^2+y^2+z^2}\sqrt{1+2^2+3^2}}$

i.e $\frac{1}{2} = \frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{14}}$

Squaring both the sides we get ,

14 $(x^2 + y^2 + z^2) = 4\,(x + 2y + 3z)^2$,

i.e 7 $(x^2 + y^2 + z^2) = 2\,(x + 2y + 3z)^2$, on simplifying we get,

$5x^2 - y^2 - 11z^2 - 8xy - 24yz - 12zx = 0$, which is req. eqn. of rt. circular cone

2. Find the eqn. of the right circular cone with vertex $(0, 0, 0)$, eqn. of the axis as $\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$ and semi vertical angle θ is 30^0.

Soln: D.R.'s of OP are x-0, y-0, z-0, and D.R's of axis are 2, -4, 3 and θ =30^0

$\therefore$ cos30 = $\frac{2x+y(-4)+z(3)}{\sqrt{x^2+y^2+z^2}\sqrt{2^2+(-4^2)+3^2}}$

i.e $\frac{\sqrt{3}}{2} = \frac{2x-4y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{29}}$

$\sqrt{3}\sqrt{x^2 + y^2 + z^2}\,\sqrt{29} = 2(2 - 4y + 3z)$

Squaring both the sides , we get

$87(x^2 + y^2 + z^2) = 4\,(2 - 4y + 3z)^2$ which is req. eqn. of rt. circular cone

3. Find the eqn. of the right circular cone with

(i) vertex $(0, 0, 0)$, eqn. of the axis as $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ and semi vertical angle θ is 45^0.

(ii) vertex $(0, 0, 0)$, eqn. of the axis as $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and semi vertical angle θ is 30.

(iii) vertex $(0, 0, 0)$, eqn. of the axis as $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$ and semi vertical angle θ is 60^0.

Try above three examples as Home work

4. Find the eqn. of the right circular cone with vertex as (0,0,0) , axis as z-axis and θ=30^0.

Soln.: We know that eqn. of the right circular cone with vertex as (0,0,0) , axis as z-axis and smi vertical angle as θ is $z^2\tan^2\theta = x^2 + y^2$

But θ = 30^0, ∴ we have $z^2\tan^2(30) = x^2 + y^2$

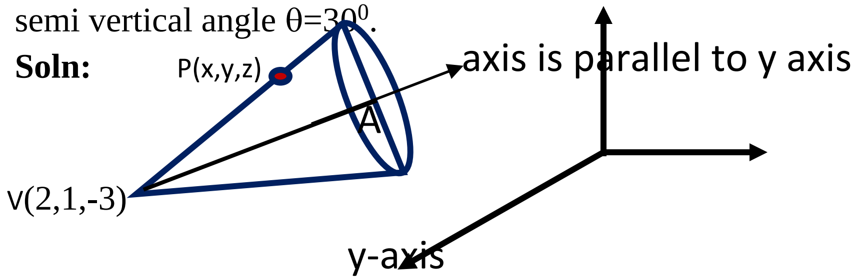
i.e $z^2\frac{1}{3} = x^2 + y^2$ => $3(x^2 + y^2) - z^2 = 0$, req. eqn. of cone.

5. Find the eqn. of the right circular cone with vertex as (0,0,0) , axis as x-axis and θ=60^0.
6. Find the eqn. of the right circular cone with vertex as (0,0,0) , axis as y-axis and θ=50^0.

Try above two examples as Home work

7. Find the eqn. of the right circular cone with vertex as (2,1,-3) , axis is parallel to y-axis and semi vertical angle θ=30^0.

Soln:



Let P(x,y,z) be any point on the surface of the cone and V(2,1,-3)be vertex of the cone

D.R.'s of VP are x-2, y-1, z+3,

Since axis of the cone is parallel to y-axis and hence DR's of axis are same as DR's of y-axis, they are 0,,1 0

∴ D.R's of axis are 0, 1, 0 and θ =60^0

∴ $\cos 30 = \frac{(x-2)0+(y-1)1+(z+3)0}{\sqrt{(x-2)^2+(y-1)^2+(z+3)^2}\sqrt{0+1+0^2}}$

i.e $\frac{\sqrt{3}}{2} = \frac{(y-1)}{\sqrt{(x-2)^2+(y-1)^2+(z+3)^2}\ 1}$ i.e $\sqrt{3}\,[\sqrt{(x-2)^2+(y-1)^2+(z+3)^2}] = 2(y-1)$

Squaring both the sides we get

$3[(x-2)^2+(y-1)^2+(z+3)^2] = 4(y-1)^2$

i.e $3(x-2)^2-(y-1)^2+3(z+3)^2=0$ which is req. eqn. of rt. circular cone.

8. Find the eqn. of the right circular cone with vertex as (2,-3, 5) , axis making equal angle with Coordinate axis and semi vertical angle θ=30^0.

Soln.:Let P(x,y,z) be any point on the surface of the cone and V(2,-3, 5)be vertex of the cone

D.R.'s of VP are x-2, y+3, z-5,

Since axis of the cone makes equal angle with coordinate axis and hence

D.R's of axis are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ & θ =30^0

$$\therefore \cos 30 = \frac{(x-2)\frac{1}{\sqrt{3}}+(\ y+3)\frac{1}{\sqrt{3}}+(z-3)\frac{1}{\sqrt{3}}}{\sqrt{(x-2)^2+(y+3)^2+(z-5)^2}\ \sqrt{\frac{1}{\sqrt{3}}^2+\frac{1}{\sqrt{3}}^2+\frac{1}{\sqrt{3}}^2}}$$

i.e $\frac{\sqrt{3}}{2} = \frac{(x-2)+(\ y+3)+(z-3)}{\sqrt{3}\sqrt{(x-2)^2+(y+3)^2+(z-5)^2}\ \sqrt{1}}$

i.e $3\ [\sqrt{(x-2)^2+(y+3)^2+(z-5)^2}] = 2[(x-2)+(\ y+3)+(z-5)]$

Squaring both the sides we get,

$9[(x-2)^2+(y+3)^2+(z-5)^2] = 4\ [(x-2)+(\ y+3)+(z-5)]^2$

i**.e 9[x^2+y^2+z^2 -4x + 6y – 10z +38] =4[x + y + z -4]^2 , which is req. eqn. of cone.**

9. Find the eqn. of the right circular cone with vertex $(1,-2,1)$, eqn. of the axis as

$\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$ and semi vertical angle θ is 60^0.

Soln: Let P(x,y,z) be any point on the surface of the cone and V(1,-2, -1)be vertex of the cone

D.R.'s of VP are x-1, y+2, z+1,

D.R's of axis are 3, -4, 5 & θ =60^0

$$\therefore \cos 60 = \frac{(x-1)3+(\ y+2)(-4)+(z+1)5}{\sqrt{(x-1)^2+(y+2)^2+(z-1)^2}\ \sqrt{3^2+(-4)^2+5^2}}$$

i.e $\frac{1}{2} = \frac{3(x-1)-4(\ y+2)+5(z+1)}{\sqrt{(x-2)^2+(y+3)^2+(z+1)^2}\ \sqrt{50}}$

i.e $\sqrt{50}\ [\sqrt{(x-2)^2+(y+3)^2+(z+1)^2}] = 2[3(x-1)-4(\ y+2)+5(z+1)]$

Squaring both the sides we get,

$50[(x-2)^2+(y+3)^2+(z+1)^2] = 4\ [3x-4y+5z-6]^2$

i**.e 25[x^2+y^2+z^2 -4x + 6y +2z +14] =2[x + y + z -4]^2 , which is req. eqn. of cone.**

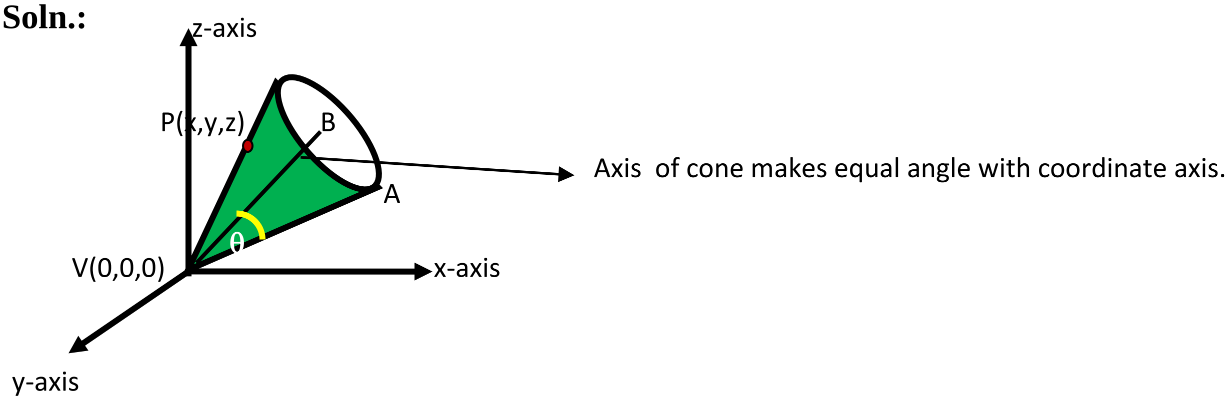
10. Find the eqn. of the right circular cone with vertex $(3,2,1)$, eqn. of the axis as

$\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$ and semi vertical angle θ is 30^0.

HOME work

11. Find the eqn. of the right circular cone with vertex origin, axis making equal angles with coordinate axis and whose generator has DR's 1, -2, 2.

Soln.:



In this example, θ is not given.

Given that , axis makes equal angles with coordinate axis and hence DR's of axis are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

And DR's of generator VA are 1, -2, 2.

We know that axis makes equal angle with all generators in rt. circular cone, ∴ θ is same throughout .

From the fig. angle between VA and VB is θ

$$\therefore \cos\theta = \frac{1\left(\frac{1}{\sqrt{3}}\right)+(-2)\frac{1}{\sqrt{3}}+2\left(\frac{1}{\sqrt{3}}\right)}{\sqrt{1^2+(-2)^2+2^2}\ \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2}} = \frac{1\left(\frac{1}{\sqrt{3}}\right)}{\sqrt{9}\ \sqrt{1}} = \frac{1}{3\sqrt{3}} \text{-----------------------------(1)}$$

Next, let P(x, y, z) be any point on the surface of the cone, again from the fig. angle between VP and VB is also θ

We have $\cos\theta = \frac{x\left(\frac{1}{\sqrt{3}}\right)+y\frac{1}{\sqrt{3}}+z\left(\frac{1}{\sqrt{3}}\right)}{\sqrt{x^2+y^2+z^2}\ \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2}}$ (b'cz, DR's of VP are x, y, z)

But from (1) $\cos\theta = \frac{1}{3\sqrt{3}}$

∴ we have $\frac{1}{3\sqrt{3}} = \frac{x+y+z}{\sqrt{3}\sqrt{x^2+y^2+z^2}\ \sqrt{1}}$

i.e $\sqrt{x^2+y^2+z^2}$ = 3(x+ y +z)

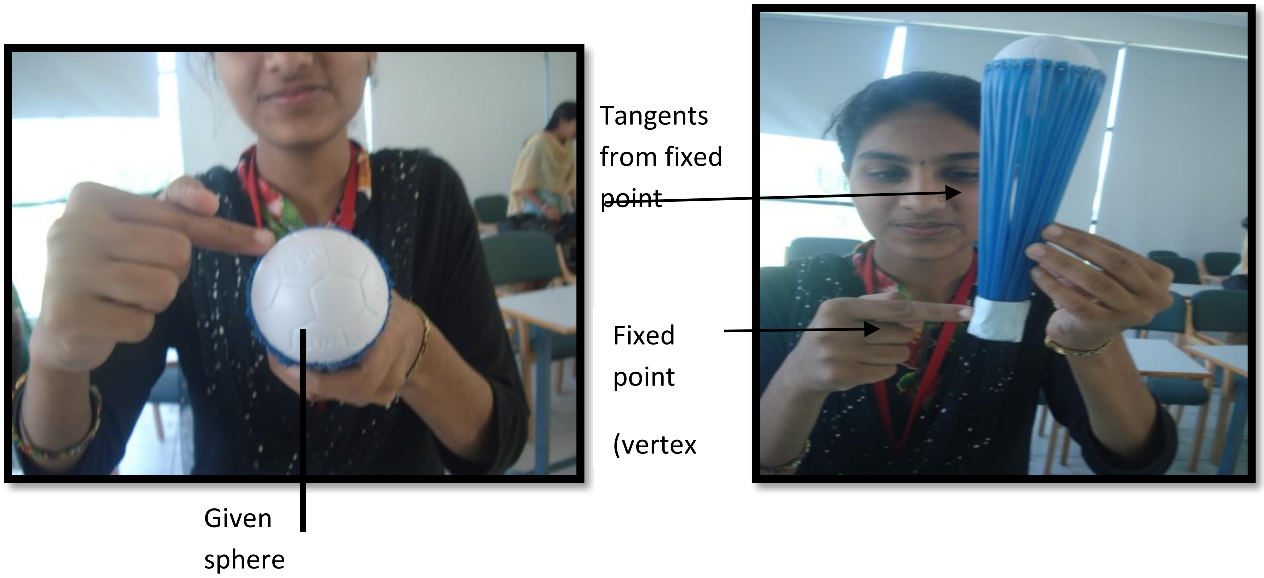
Squaring both the sides we get, $[x^2+y^2+z^2]$ = 9(x+ y +z)^2

i.e.8$(x^2+y^2+z^2)$ +18(xy+yz+zx) = 0

i.e 4$(x^2+y^2+z^2)$ +9(xy+yz+zx) = 0 which is req. eqn. of rt. circular cone.

Enveloping cone of a sphere:

We know that from the external point to the surface of the sphere if we draw tangents throughout



the sphere i.e go on drawing tangent lines as much as possible to sphere from the fixed point, we get one surface which is in the form of cone, that is called enveloping cone of sphere.

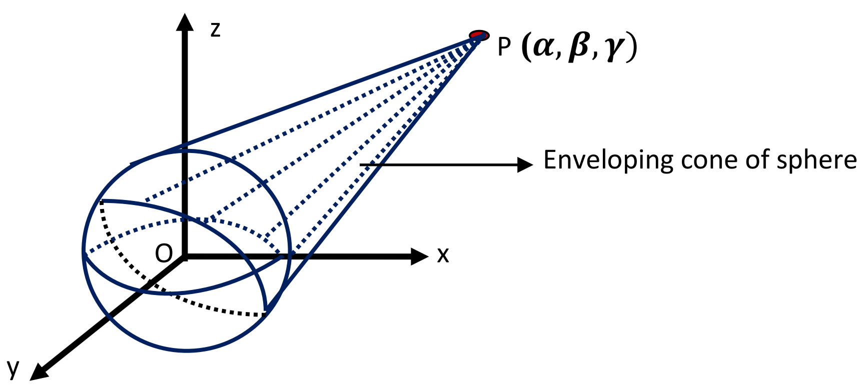
$\therefore$ we define enveloping cone of sphere as follows

Defn: The locus of tangent lines drawn from a given point to a given sphere is called enveloping cone of the sphere. The given point is vertex of the cone.

Note: Defn. is important for two marks**.**

Theorem: Find the eqn. of enveloping cone of sphere $x^2 + y^2+z^2 =a^2$, from the vertex $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$.

Proof:



Give sphere is $\mathbf{x^2 + y^2+z^2 =a^2}$-- **(1)**

Any line through P$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ **with DR's l,m,n is** $\frac{x-\boldsymbol{\alpha}}{l} = \frac{y-\boldsymbol{\beta}}{m} = \frac{z-\boldsymbol{\gamma}}{n}$ =r (say)---------------(2)

$\therefore$ any point on the line (2) is $(\alpha + lr, \beta + mr, \gamma + nr)$

It lies on the sphere (1) if it satisfies eqn. (1).

i.e $(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$

i.e $r^2(l^2+m^2+n^2) + r(2\alpha l + 2\beta m + 2\gamma n) + (\alpha^2+\beta^2+\gamma^2 - a^2) = 0$---------------------------(3)

which is quadratic in r, has two roots for r. For two different values of r we get two points at which a line will intersect sphere, i.e any line will intersect sphere at two points, but if it is a tangent then it will touch the sphere at only one point, hence both values of r same. Condition for equal roots of r discriminant $b^2 - 4ac = 0$ in (3)

i.e $(2\alpha l + 2\beta m + 2\gamma n)^2 - 4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2 - a^2) = 0$

i.e $(\alpha l + \beta m + \gamma n)^2 - (l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2 - a^2) = 0$-----------------------------------(4)

From (2), $l = \frac{x-\alpha}{r}$, $m = \frac{y-\beta}{r}$, $n = \frac{z-\gamma}{r}$, substitute these values in (4) we get

$(\alpha \frac{x-\alpha}{r} + \beta \frac{y-\beta}{r} + \gamma,)^2 - ((\frac{x-\alpha}{r})^2+(\frac{y-\beta}{r})^2+(\frac{z-\gamma}{r})^2)(\alpha^2+\beta^2+\gamma^2 - a^2) = 0$

i.e $[\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma)]^2 - [(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2](\alpha^2+\beta^2+\gamma^2 - a^2) = 0$

By simplifying we get, $[\alpha x + \beta y + \gamma z - a^2]^2 - [x^2+y^2+z^2 - a^2][\alpha^2+\beta^2+\gamma^2 - a^2] = 0$

i.e $\boldsymbol{[\alpha x + \beta y + \gamma z - a^2]^2 = [x^2+y^2+z^2 - a^2][\alpha^2+\beta^2+\gamma^2 - a^2]}$ ---(5)

which is req. eqn. of enveloping cone of sphere.

Easy method to remember this eqn. is as follows:

Eqn. (5) can be written as, $\mathbf{T^2 = SS_1}$

where $T = \alpha x + \beta y + \gamma z - a^2$ i.e eqn. of tangent to the sphere at (α, β, γ)

$S = x^2 + y^2+z^2 - a^2$ i.e given sphere

$S_1 = \alpha^2+\beta^2+\gamma^2 - a^2$ i.e Sphere at (α, β, γ)

Note: (1) Above formula $\mathbf{T^2 = SS_1}$, u have to remember for examples.

(2) Sphere may be given not necessarily with centre at origin, it may given general eqn. **$x^2 + y^2+z^2+2ux+2vy+2wz+d = 0$, then also same formula $T^2 = SS_1$, but**

T $= \alpha x + \beta y + \gamma z + u(x+\alpha) + v(y+\beta) + w(z+\gamma) + d$ i.e Eqn. of tgt. to this sphere

$S = x^2 + y^2+z^2+2ux+2vy+2wz+d$, i.e eqn. of given sphere

$S_1 = \alpha^2+\beta^2+\gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d$, i.e sphere at (α, β, γ)

Note: Enveloping cone of the sphere is also locus of all tangents drawn from point (($\boldsymbol{\alpha, \beta, \gamma}$) to sphere.

Examples:

1. Find the enveloping cone of the sphere $x^2 + y^2+z^2 = 11$ which has the vertex (2,4,1). Also prove that the plane z=0 cuts this cone in rectangular hyperbola.

Soln.: Given sphere $S = x^2 + y^2+z^2 - 11$ and $(\alpha, \beta, \gamma) = (2,4,1)$

$\therefore \quad S_1 = 2^2 + 4^2+1^2 - 11 = 10$

$T = 2x+4y+z-11$

Enveloping cone of sphere is $\mathbf{T^2 = SS_1}$

i.e $(2x+4y+z-11)^2 = (x^2 + y^2+z^2 - 11)(10)$

i.e $10(x^2 + y^2+z^2 - 11) = (2x+4y+z-11)^2$, req. eqn. of enveloping cone of sphere.

Next, If plane z=0 cuts this cone, put z=0 in above eqn. we get,

$10(x^2 + y^2 - 11) = (2x+4y -11)^2$

i.e $6x^2 + 6y^2 - 16xy +44x -88y - 231 =0$, which is rectangular hyperbola.

2. Find the enveloping cone of the sphere $x^2 + y^2+z^2 = 9$ which has the vertex (0,1,1).

Soln.: Given sphere $S = x^2 + y^2+z^2 - 9=0$ and $(\alpha, \beta, \gamma) =(0,1,1)$

$\therefore \quad S_1= 0^2 + 1^2+1^2 - 9 = -7$

$T = 0x+1y+1z-9 = y+z-9$

Enveloping cone of sphere is $\mathbf{T^2 = SS_1}$

i.e $(y+z-9)^2 = (x^2 + y^2+z^2 - 9)(-7)$

i.e $7(x^2 + y^2+z^2 - 9) + (y+z-9)^2$, req. eqn. of enveloping cone of sphere.

3. Find the enveloping cone of the sphere $x^2 + y^2+z^2 -2x +4z-1=0$ with the vertex (1,1,1).

Soln.: Given sphere $S = x^2 + y^2+z^2 - 2x+4z-1 =0$ and $(\alpha, \beta, \gamma) =(1,1,1)$

$\therefore \quad S_1= 1^2 + 1^2+1^2 - 2+4 -1 = 4$

$T = \alpha x + \beta y + \gamma z + u(x+ \alpha) + v(y+ \beta) + w(z+ \gamma) + d$

$= x+y+z-1(x+1)+0(y+1)+2(z+1)-1 = y+3z$

Enveloping cone of sphere is $\mathbf{T^2 = SS_1}$

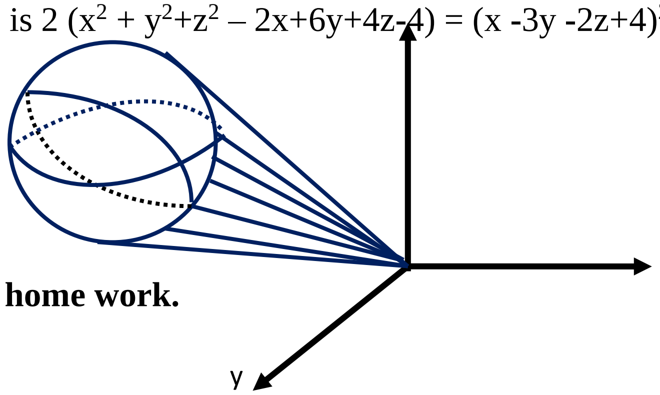
i.e $(y+3z)^2 = (x^2 + y^2+z^2 - 2x+4z-1)(4)$

i.e $4(x^2 + y^2+z^2 - 2x+4z-1) = (y+3z)^2$, req. eqn. of enveloping cone of sphere.

4. Find the enveloping cone of the sphere $x^2 + y^2+z^2 +2x -2y=2$ with the vertex (1,1,1).

5. Prove that the lines drawn from the origin so as to touch the sphere $x^2 + y^2+z^2 - 2x+6y+4z-4=0$ lie on the cone $-4 (x^2 + y^2+z^2 - 2x+6y+4z-4) = (x -3y -2z+4)^2$

Hint: In this example you have to prove enveloping cone of sphere $x^2 + y^2+z^2 - 2x+6y+4z-4=0$ From vertex (0,0,0) is $2 (x^2 + y^2+z^2 - 2x+6y+4z-4) = (x -3y -2z+4)^2$



Try (4) and (5) as home work.

Try following examples also:

1. Prove that equation (i) $4x^2 - y^2 +2z^2+2xy - 3yz +12x - 11y +6z +4=0$ represents cone with vertex (-1, -2, -3).

UNIT-V

CYLINDER

Syllabus:

Unit – V

Cylinder

Equation of a cylinder, Enveloping cylinder of a Sphere, Right circular cylinder.

-10HRS

K.L.E Society's

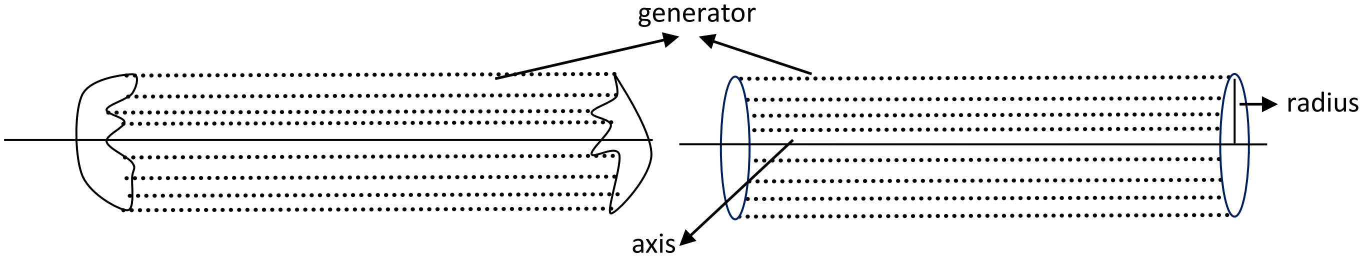
G.I. Bagewadi Arts, Science & Commerce College, NIPANI

DEPARTMENT OF MATHEMATICS

B.Sc. II Sem. Paper II

Lecture on Cylinder April 2020

Definition: A surface generated by variable straight line moving in space parallel to a fixed line and satisfying the condition that interesting given curve is called cylinder. Variable line is called generator and intersecting curve is called guiding curve.



Oblique cylinder **Right circular cylinder**

Right circular cylinder: A surface generated by variable straight line moving in space parallel to a fixed line with constant distance from fixed line and satisfying the condition that interesting given curve is called cylinder. Fixed line is called axis of the cylinder, variable line is called generator, constant distance is called radius of cylinder and intersecting curve is called guiding curve.

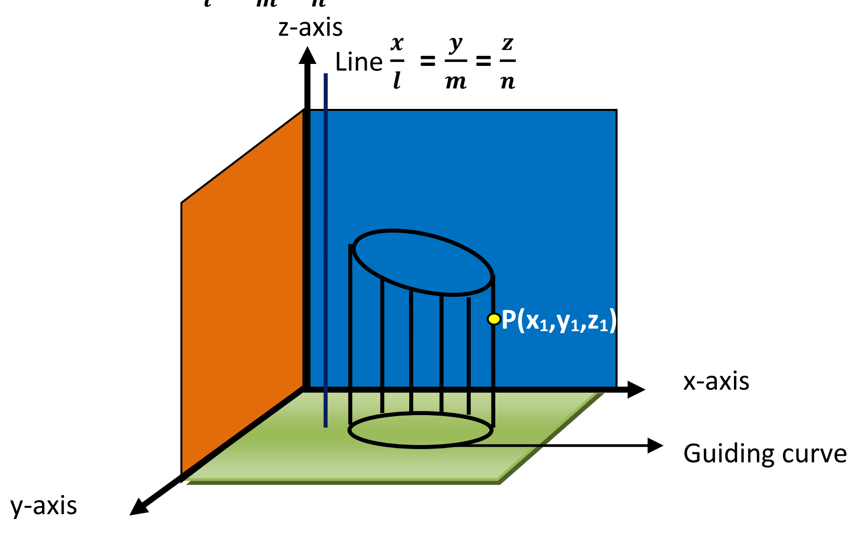
Note:(1) In oblique cylinder generators are not necessarily at constant distance from fixed line where as in rt. circular cylinder they are at constant distance.

(2) Most of the properties and theorems in cylinder are same as that of cone.

(3) It is not having vertex, only fixed line and conic

Theorem: To find eqn. of cylinder with given conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, $z=0$ and generators parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Proof:



Given conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, z=0 ----------------------(1)

Let the equation of the fixed line be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ --------------------(2)

Let $P(x_1,y_1,z_1)$ be any point on the cylinder, then equation to this line parallel to fixed line (2) is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ ---------------------------------------(3)

But this line intersect xy-plane at z=0 (b'cz on xy plane z coordinate is zero)

∴ we have $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{0-z_1}{n}$---**(4)**

⇨ $\frac{x-x_1}{l} = \frac{0-z_1}{n}$ and $\frac{y-y_1}{m} = \frac{0-z_1}{n}$

⇨ $x = \boldsymbol{x_1} - \frac{lz_1}{n}$, $y = \boldsymbol{y_1} - \frac{mz_1}{n}$, z = 0,

But line (4) intersect the given conic (1) if this point lies on conic (1), as shown in fig. ∴ substitute this point in (1) we get ,

$a(x_1 - \frac{lz_1}{n})^2 + 2h(x_1 - \frac{lz_1}{n})(y_1 - \frac{mz_1}{n}) + b(y_1 - \frac{mz_1}{n})^2 + 2g(x_1 - \frac{lz_1}{n}) + 2f(y_1 - \frac{mz_1}{n}) + c = 0$

(not necessary to remove l, m,n as they are given)

Simplifying this eqn. we get

$a(nx_1 - lz_1)^2 + 2h(nx_1 - lz_1)(ny_1 - mz_1) + b(nx_1 - lz_1)^2 + 2g(nx_1 - lz_1) + 2f(nx_1 - lz_1) + cn^2 = 0$

Then locus of $P(x_1, y_1, z_1)$, i.e replace (x, y, z) we get,

$a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(nx - lz)^2 + 2g(nx - lz) + 2f(nx_1 - lz_1) + cn^2 = 0.$

Thus required eqn. of cylinder with generators parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and given conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, z=0 (i.e guiding curve) is

$a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(nx - lz)^2 + 2g(nx - lz) + 2f(nx_1 - lz_1) + cn^2 = 0$.----(5)

Corollary 1. If the generator of the cylinder parallel to z-axis then l=0, m=0 and n=1 substitute these in (5) we get eqn. of cylinder is **$ax^2+2hxy+by^2+2gx+2fy+c=0$, z=0**
i.e f(x,y) =0, z=0

similarly, If the generator of the cylinder parallel to x-axis then l=1,m=0 and n=0 substitute these in (5) we get eqn. of cylinder is **f(z,y) =0, x=0.**

And If the generator of the cylinder parallel to y-axis then l=0,m=1 and n=0 substitute these in (5) we get eqn. of cylinder is **f(x,z) =0, y=0.**

Corollary 2. The eqn. of cylinder which is having guiding curve as intersection of two curves f(x,y,z) =0----(1) and g(x, y, z)=0-----(2) and whose generators are parallel to

(i) x –axis is obtained by eliminating x between (1) and (2)
(ii) y–axis is obtained by eliminating y between (1) and (2)
(iii) z–axis is obtained by eliminating z between (1) and (2)

Examples:

1. Find eqn. to the cylinder which passes through the curve of intersection of plane lx+my+nz=p and $ax^2+by^2+cz^2 = 1$ and generators parallel to z-axis.

Soln. Given guiding curve is intersection of lx+my+nz=p ---------------------(1)

$ax^2+by^2+cz^2 = 1$ ----------------------(2)

Since generators are parallel to z-axis, req. eqn. to the cylinder is obtained by eliminating z-coordinate between (1) and (2)

From (1) $z = \frac{p-lx-my}{n}$, substitute this in (1) we get

$ax^2+by^2+c(\frac{p-lx-my}{n})^2 = 1$

i.e **$an^2x^2+bn^2y^2+c(p-lx-my)^2 = n^2$, which is the req. eqn. of the cylinder.**

2. Find eqn. to the cylinder which passes through the curve of intersection of plane lx+my+nz=p and $by^2+cz^2 = 2ax$ and generators parallel to x -axis.

Hint: get x from first eqn. and substitute in second as in above example.

3. Find eqn. to the cylinder which passes through the curve of intersection of plane lx+my+nz=p and $ax^2 +cz^2 = 2y$ and generators parallel to y -axis.

Hint: get y from first eqn. and substitute in second as in above example.

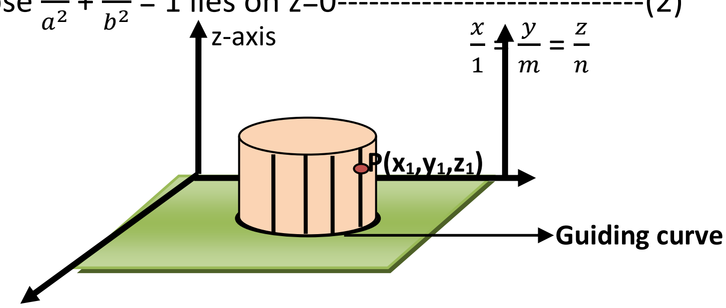
Try (2)and (3) as home work.

4. Find eqn. to the surface generated by the line which is parallel to y=mx and z=nx and intersecting the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, z=0

i.e nothing but , finding eqn. of cylinder whose generators are parallel to the line y=mx and z=nx and intersecting the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ lies on z=0 i.e xy plane

Soln: Given line y=mx and z=nx i.e $x = \frac{y}{m} = \frac{z}{n}$, i.e $\frac{x}{1} = \frac{y}{m} = \frac{z}{n}$ ----------------(1)

Guiding curve is ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ lies on z=0-----------------------------(2)



Let $P(x_1,y_1,z_1)$ be any point on the cylinder, then equation generator parallel to

given line (1) passing through is $\frac{x-x_1}{1} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ **--(3)**

But this line intersect xy-plane at z=0 (b'cz on xy plane z coordinate is zero)

$\therefore$ we have $\frac{x-x_1}{1} = \frac{y-y_1}{m} = \frac{0-z_1}{n}$ **--(4)**

⇨ $\frac{x-x_1}{1} = \frac{0-z_1}{n}$ and $\frac{y-y_1}{m} = \frac{0-z_1}{n}$

⇨ $x = x_1 - \frac{z_1}{n}$, $y = y_1 - \frac{mz_1}{n}$, z = 0,

But line (4) intersect the given conic (2) if this point lies on conic (2), as shown in fig. $\therefore$ substitute this point in (2) we get ,

$$\frac{(x_1 - \frac{z_1}{n})^2}{a^2} + \frac{(y_1 - \frac{mz_1}{n})^2}{b^2} = 1$$

i.e $b^2(x_1 - \frac{z_1}{n})^2 + a^2(y_1 - \frac{mz_1}{n})^2 = a^2b^2$

i.e $b^2(nx_1 - z_1)^2 + a^2(ny_1 - z_1)^2 = a^2b^2n^2$

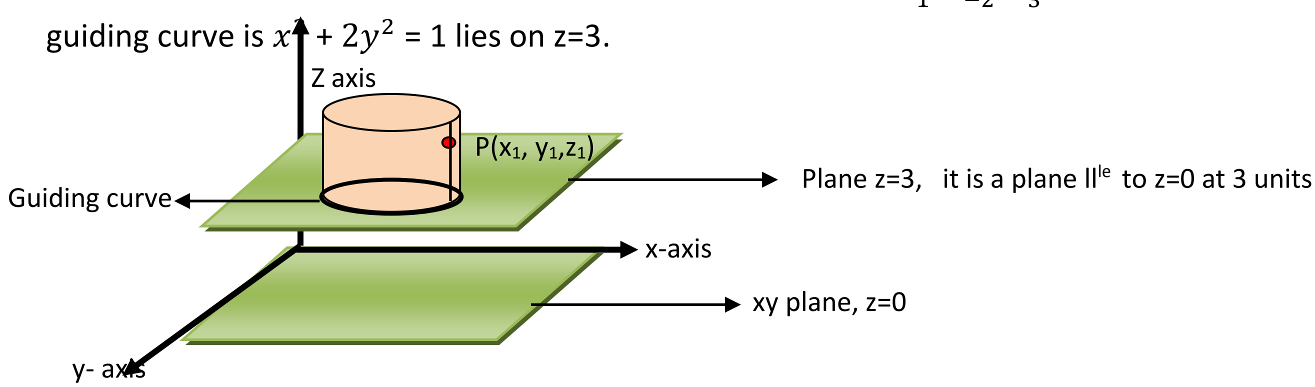
Locus of $P(x_1,y_1,z_1)$ i.e replace (x_1,y_1,z_1) by (x, y, z) we get

$b^2(nx - z)^2 + a^2(ny - z)^2 = a^2b^2n^2$**, req. Eqn. of cylinder**.

5. Find eqn. of cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ and whose guiding curve is $x^2 + 2y^2 = 1$ lies on z=0.

Try this example, similar to above example.

6.Find eqn. of cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + 2y^2 = 1$ lies on z=3.



Soln.:Gven line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ ---------------------(1) guiding curve is $x^2 + 2y^2 = 1$, z=3-----(2)

Let $P(x_1,y_1,z_1)$ be any point on the cylinder, then equation of generator parallel to given line (1) passing through is $\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$ **--(3)**

But this line intersect plane z=3 (plane II[le] to xy plane at distance z=3)

$\therefore$ we have $\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{3-z_1}{3}$---**(4)**

⇨ $\frac{x-x_1}{1} = \frac{3-z_1}{3}$ and $\frac{y-y_1}{-2} = \frac{3-z_1}{3}$

⇨ $(3x - 3x_1) = 3 - z_1$, $3(y - y_1) = -2(3 - z_1)$, z = 3

⇨ x = $\frac{3x_1+3-z_1}{3}$, y = $\frac{3y_1-6-2z_1}{3}$, z=3

If the line (4) intersect the given guiding curve, the above point satisfies eqn. (2)

$\therefore$ we have $\left(\frac{3x_1+3-z_1}{3}\right)^2 + 2\left(\frac{3y_1-6-2z_1}{3}\right)^2 = 1$

i.e $(3x_1 + 3 - z_1)^2 + 2(3y_1 - 6 - 2z_1)^2 = 9$

$\therefore$ locus of this eqn. is $(3x - z + 3)^2 + 2(3y - 2z - 6)^2 = 9$, which is the req. eqn.of cylinder.

7. Find eqn. of cylinder whose guiding curve is $x^2 + y^2 = 16$ lies on z=0 and generators are parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.

Soln.: Given line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ --(1)

Guiding curve is ellipse $x^2 + y^2 = 16$ lies on z=0----------------------------(2)

Let P(x_1,y_1,z_1) be any point on the cylinder, then equation generator parallel to given line (1) passing through is $\frac{x-x_1}{1} = \frac{y-y_1}{2} = \frac{z-z_1}{3}$ --(3)

But this line intersect xy-plane at z=0 (b'cz on xy plane z coordinate is zero)

$\therefore$ we have $\frac{x-x_1}{1} = \frac{y-y_1}{2} = \frac{0-z_1}{3}$---**(4)**

⇨ $\frac{x-x_1}{1} = \frac{0-z_1}{3}$ and $\frac{y-y_1}{2} = \frac{0-z_1}{3}$

⇨ $x = x_1 - \frac{z_1}{3}$, $y = y_1 - \frac{2z_1}{3}$, z = 0,

But line (4) intersect the given conic (2) if this point lies on conic (2), as shown in fig. $\therefore$ substitute this point in (2) we get ,

$\left(x_1 - \frac{z_1}{3}\right)^2 + \left(y_1 - \frac{2z_1}{3}\right)^2 = 16$

i.e $(3x_1 - z_1)^2 + (3y_1 - 2z_1)^2 = 16$

$\therefore$ locus of this eqn. is $(3x - z)^2 + (3y - 2z)^2 = 16$

i.e $9x^2 + 9y^2 + 10z^2 - 6xz - 12yz - 16 = 0$ req. eqn. of cylinder.

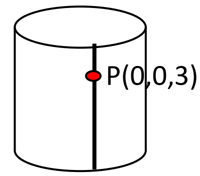
8. Find eqn. of cylinder whose generators are parallel to the line $x = \frac{y}{2} = -z$,

i.e $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and passing through the curve (i.e. guiding curve) $3x^2 + 2y^2 = 1$,z=0.

Try this example as home work.

9. Show that the line $\frac{x}{1} = \frac{y}{-1} = \frac{z-3}{1}$ is a generator of the cylinder **$x^2 + y^2 + z^2 + xy + yz - zx = 9$.**

Soln.: Given line $\frac{x}{1} = \frac{y}{-1} = \frac{z-3}{1}$ i.e $\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-3}{1}$-------------------(1)



i.e the line passing through point P(0,0,3)

And given cylinder $\mathbf{x^2 + y^2 + z^2 + xy + yz - zx = 9}$ **-------------------------(2),**

To prove the line (1) is generator for the cylinder (2)we have to prove that(0,0,3) lies on (2), i.e this point satisfies (2).

i.e 0+0+9+0+0+0=9

9= 9

⇨ **(1) is the generator of the cylinder (2).**

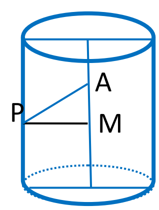
10. Find eqn. of cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $z^2 + y^2 = 1$ lies on x=2.

Right circular cylinder:

We know that **right circular cylinder** is a surface generated by variable straight line parallel to a fixed line with constant distance from fixed line and satisfying the condition that interesting given curve, that curve is circle.

Finding equation of the right circular cylinder whose radius is r and axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ **(important for 5 marks)**

Slon:



Let (x_1, y_1, z_1) be any point on the cylinder and let $A(\alpha, \beta, \gamma)$ be a fixed point on the axis and PM = r, radius of the cylinder.

Then from the fig. $AP^2 = PM^2 + AM^2$

⇨ $PM^2 = AP^2 - AM^2$ --------------------------(1)

By using distance formula, AP = $\sqrt{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2}$

AM = Projection of AP on the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

AM = $\frac{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n}{\sqrt{l^2+m^2+n^2}}$ (By using dot product)

Squaring both the sides we get,

$AM^2 = \left(\frac{(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n}{\sqrt{l^2+m^2+n^2}}\right)^2 = \frac{[(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n]^2}{(l^2+m^2+n^2)}$ and PM =r

Then (1) becomes $r^2 = [(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2] - \frac{[(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n]^2}{(l^2+m^2+n^2)}$

$$r^2((l^2+m^2+n^2) = [(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2](l^2+m^2+n^2) - [(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n]^2$$

i.e $[(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n]^2 = [(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2 - r^2](l^2+m^2+n^2)$

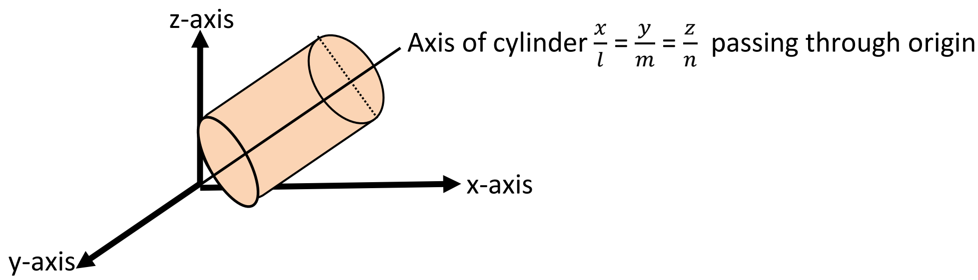
Locus of the above eqn. is

$$[(\mathbf{x}-\boldsymbol{\alpha})\mathbf{l}+(\mathbf{y}-\boldsymbol{\beta})\mathbf{m}+(\mathbf{z}-\boldsymbol{\gamma})\mathbf{n}]^2 = [(\boldsymbol{x}-\boldsymbol{\alpha})^2+(\boldsymbol{y}-\boldsymbol{\beta})^2+(\boldsymbol{z}-\boldsymbol{\gamma})^2] - \mathbf{r}^2\,](\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)$$

Which is the required eqn. of the rt. circular cylinder with axis and radius r.

Corollary 1. Eqn. of the rt. circular cylinder with axis $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and radius r is $[lx+my+nz]^2 = [x^2+y^2+z^2-r^2][l^2+m^2+n^2]$

Proof:



We know that eqn. of cylinder with axis $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and radius as r is

$[(x-\alpha)l+(y-\beta)m+(z-\gamma)n]^2 = [(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2] - r^2\,](l^2+m^2+n^2)$

$\therefore$ eqn. of cylinder with axis $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (i.e line passing through origin with DR's l, m, n) and radius as r is $[xl+ym+zn]^2 = [x^2+y^2+z^2-r^2](l^2+m^2+n^2)$

i.e $[lx+my+nz]^2 = [x^2+y^2+z^2-r^2][l^2+m^2+n^2]$

Corollary 2. Eqn. of the rt. circular cylinder with radius r and (i) z –axis as axis of cylinder is $x^2+y^2=r^2$ (ii) y –axis as axis of cylinder is $x^2+z^2=r^2$ (iii) x –axis as axis of cylinder is $y^2+z^2=r^2$

Proof:

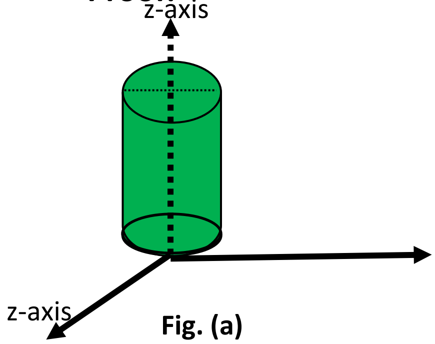


Fig. (a)

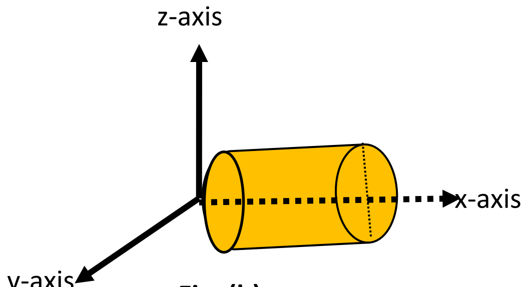


Fig. (b)

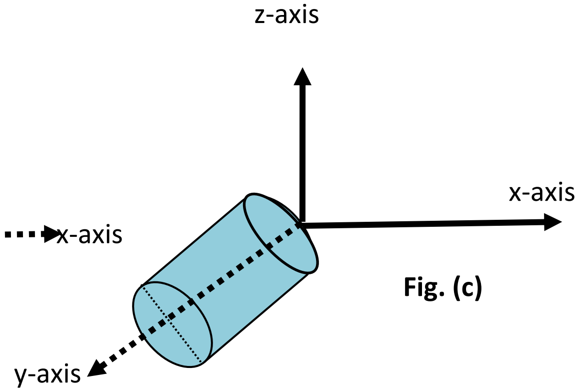


Fig. (c)

We know that eqn. of rt. circular cylinder with axis $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and radius as r is

$[lx + my + nz]^2 = [x^2 + y^2 + z^2 - r^2][l^2 + m^2 + n^2]$ by **cor. (1)--------------------(d)**

If z-axis is axis of cylinder as in fig. (a), then $l = 0$, m = 0, n = 1

$\therefore$ eqn. (d) becomes $[0 + 0 + z]^2 = [x^2 + y^2 + z^2 - r^2][0 + 0 + 1^2]$

i.e $z^2 = x^2 + y^2 + z^2 - r^2 \quad => x^2 + y^2 = r^2$

$\therefore$ eqn. of rt. circular cylinder with $z - axis$ as axis and radius as r is $\mathbf{x^2 + y^2 = r^2}$

Similarly, eqn. of rt. circular cylinder with $x - axis$ as axis and radius as r is $\mathbf{z^2 + y^2 = r^2}$ $\mathbf{fig. (b)}$

And eqn. of cylinder with $y - axis$ as axis and radius as r is $\mathbf{x^2 + z^2 = r^2}$ **fig. (c)**

Examples:

1. **Find the eqn. of right circular cylinder of radius 2 and axis is the line** $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$.

Soln: Axis of the cylinder $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1} = \frac{x-\alpha}{2} = \frac{y-\beta}{2} = \frac{z-\gamma}{-1}$ $\therefore$ $(\alpha, \beta, \gamma) = (1,3,5)$,

l=2, m= 2 , n= -1, radius r =2

We know that eqn. of rt. circular cylinder with axis $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and radius as r is

$[(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2 = [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2](l^2 + m^2 + n^2)$

i.e $[(x-1)2 + (y-3)2 + (z-5)(-1)]^2 = [(x-1)^2 + (y-3)^2 + (z-5)^2 - 2^2](4 + 2 + 1)$

i.e $[2x + 2y - z - 3]^2 = 9[(x-1)^2 + (y-3)^2 + (z-5)^2 - 2^2]$ req. . eqn of rt. circular cylinder.

2. Find the eqn. of right circular cylinder of radius 2 and axis is the line x =2y = -z

Soln.: eqn. of rt. circular cylinder with axis **x =2y = -z** **i.e x =2y = -z i.e** $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ and radius as r = 2, is

$[(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2 = [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2](l^2 + m^2 + n^2)$

$(\alpha, \beta, \gamma) = (0,0,0)$

i.e $[(x)2 + (y)1 + z(-2)]^2 = [(x)^2 + (y)^2 + (z)^2 - 2^2](4 + 1 + 4)$

i.e $[2x + y - z]^2 = 9[x^2 + y^2 + z^2 - 4]$

i.e $5x^2 + 8y^2 + 8z^2 + 4xy - 2yz - 4zx = 6x^2 + 6y^2 + 6z^2 - 36$

i.e $5x^2 + 8y^2 + 8z^2 - 4xy + 2yz + 4zx - 36 = 0$

3. Find the eqn. of right circular cylinder of radius 3 and axis passes through (2, 3, 4) and D.C's proportional to 2,1,-2.

Soln.: Given that axis passes through (2, 3, 4) with D.R.'s 2, 1, -2. ie e $\frac{x-2}{2} = \frac{y-3}{1} = \frac{z-4}{-2}$

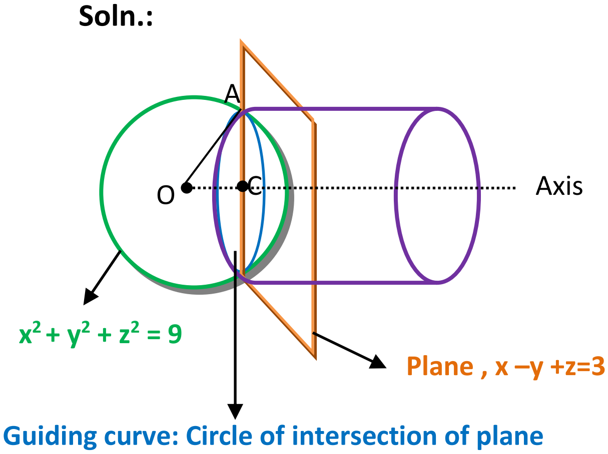
$\therefore (\alpha, \beta, \gamma) = (2, 3, 4)$ and r =3

4. Find the eqn. of right circular cylinder of radius 2 and axis e $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$

Try these two example (3) and (4).

5. Find the eqn. of rt. circular cylinder whose guiding curve is circle $x^2 + y^2 + z^2 = 9$: $x - y + z = 3$.

Soln.:



Given that circle of intersection of sphere S: $x^2 + y^2 + z^2 - 9 = 0$-- (1) and plane P: $x - y + z = 3$.---(2)
We studied in sphere that intersection of sphere by the plane is circle, that circle is the guiding curve for cylinder.

In the following fig. O(0,0,0) is centre of the sphere (1) and C is the centre of the circle of intersection, radius AC f circle is radius of the cylinder. The line joining OC is the axis of the cylinder.

So we have to find O, C, OA, AC and D.R.'s of OC.

By using the procedure of finding centre and radius of circle of intersection of sphere (1) by the plane (2) we have to find.

Centre of the sphere (1) is O = (0,0,0) and radius of the sphere OA = 3

And OC is perpendicular drawn from O to the plane (2)

$\therefore$ OC = $\left|\frac{1(0)-1(0)+1(0)-3)}{\sqrt{1^1+(-1)^2+1^2}}\right| = \left|\frac{-3)}{\sqrt{3}}\right| = \sqrt{3}$

$\therefore$ from rt. angled triangle OAC, $AC^2 = OA^2 - OC^2 = 9 - 3 = 6$

Radius of the circle i.e radius of rt. circular cylinder , AC = r $=\sqrt{6}$

And D.R.'s of axis OC are 1, -1, 1 and axis passing through $(\alpha, \beta, \gamma) = (0,0,0)$

$\therefore$ eqn. of rt. circular cylinder with radius r$=\sqrt{6}$, $(\alpha, \beta, \gamma) = (0,0,0)$, and l, m, n as 1, -1, 1

is $[lx + my + nz]^2 = [x^2 + y^2 + z^2 - r^2][l^2 + m^2 + n^2]$

i.e $[(1)x + (-1)y + (1)z]^2 = [x^2 + y^2 + z^2 - 6][1 + 1 + 1]$

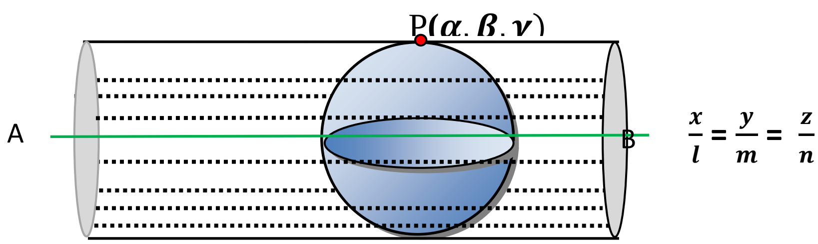
i.e $[x - y + z]^2 = 3[x^2 + y^2 + z^2 - 6]$

i.e $x^2 + y^2 + z^2 - 2xy - 2yz + 2xz = 3(x^2 + y^2 + z^2) - 18$

i.e $2(x^2 + y^2 + z^2) + 2xy + 2yz - 2xz - 18 = 0$, req. eqn. of cylinder.

Enveloping cylinder of sphere:

If we go on drawing tangent lines to the sphere parallel to the line AB, those tangent lines envelope (cover)sphere and form a surface in the form of cylinder as show in the



above figure. That cylinder formed is called enveloping cylinder of sphere. So we define enveloping cylinder as follows:

Definition: The locus of tangent lines drawn to a given sphere parallel to given line is called the enveloping cylinder of sphere.

Theorem: Enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ with tangents drawn parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is $[lx + my + nz]^2 = [x^2 + y^2 + z^2 - a^2][l^2 + m^2 + n^2 - a^2]$

Proof: Given sphere is $x^2 + y^2 + z^2 = a^2$ --(1)

Given line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ ---(2)

Let $P(\alpha, \beta, \gamma)$ be any point on the locus, any line through $P(\alpha, \beta, \gamma)$ parellel to (2) is

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ =r (say)--(3)

$\therefore$ any point on the line (3) is $(\alpha + lr, \beta + mr, \gamma + nr)$, It lies on the sphere (1) if it satisfies eqn. (1).

i.e $(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$

i.e $r^2(l^2+m^2+n^2) + r(2\alpha l + 2\beta m + 2\gamma n) + (\alpha^2+\beta^2+\gamma^2 - a^2) = 0$---------------------------**(4)**

which is quadratic in r, has two roots for r. For two different values of r we get two points at which a line will intersect sphere, i.e any line will intersect sphere at two points, but if it is a tangent then it will touch the sphere at only one point, hence both values of r same.

Condition for equal roots of r discriminant b^2 -4ac =0 in (3)

i.e $(2\alpha l + 2\beta m + 2\gamma n)^2 - 4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2 - a^2) = 0$

i.e $(\alpha l + \beta m + \gamma n)^2 - (l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2 - a^2) = 0$----------------------------------(5)

Hence, locus of (α, β, γ) i.e replace (α, β, γ) by (x, y, z) in (5) we get
$$(xl + ym + zn)^2 - (l^2+m^2+n^2)(x^2+y^2+z^2 - a^2) = 0$$
i.e. $$(lx + my + nz)^2 - (l^2+m^2+n^2)(x^2+y^2+z^2 - a^2) = 0$$
which is req. eqn. of enveloping cylinder of sphere.

Note: (i)Procedure for enveloping cone and cylinder are same , only change is, in cone tangents are drawn from a fixed point to the sphere and in cylinder tangents are drawn parallel to the given line.

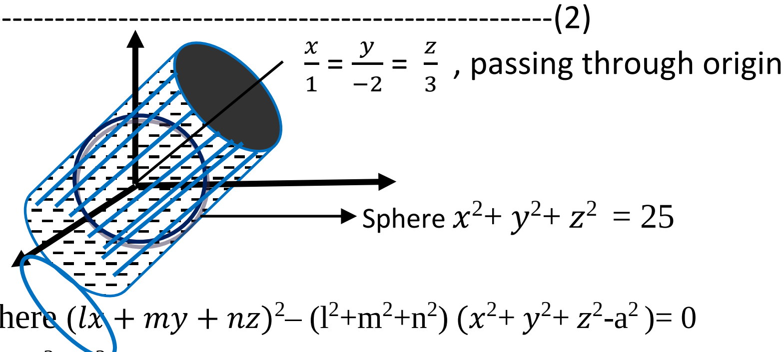
(ii) Enveloping cylinder of the sphere is also sometimes is locus of generators touch the sphere and parallel to the given line.

Examples:

1.Find the equation of the enveloping cylinder of the sphere $x^2+y^2+z^2 = 25$ whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$

Soln.: Given sphere $x^2+y^2+z^2 = 25$ --(1)

Given line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ --(2)



Here a =5, l =1, m = --2 , n =3

We know that enveloping cone of sphere $(lx + my + nz)^2 - (l^2+m^2+n^2)(x^2+y^2+z^2-a^2) = 0$

i.e $((1)x + (-2)y + 3z)^2 - (1+4+9)(x^2+y^2+z^2-25)$

i.e $(x - 2y + 3z)^2 = 14(x^2 + y^2 + z^2 - 25)$ which is req. eqn. of enveloping cone of sphere.

2. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ whosegenerators are parallel to the line $x = y = z$

Soln.: Given sphere $x^2 + y^2 + z^2 = a^2$ ---(1)

Given line $x = y = z$ i.e $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ ---(2)

Here a =a, l =1, m = 1 , n =1

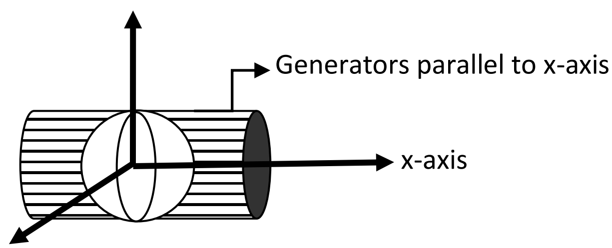
We know that enveloping cone of sphere $(lx + my + nz)^2 - (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2) = 0$

i.e $(x + y + z)^2 - (1+1+1)(x^2 + y^2 + z^2 - a^2)$

i.e $(x + y + z)^2 = 3(x^2 + y^2 + z^2 - a^2)$ which is req. eqn. of enveloping cone of sphere.

3. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ whose generators are parallel x-axis.

Soln.:



Generators parallel to x-axis => D.R.'s of generators are 1,0,0

$\therefore$ put l=1, m=0, n=0 in above eqn. we get $x^2 = (x^2 + y^2 + z^2 - a^2)$

i.e $y^2 + z^2 = a^2$

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